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On the recovering of a coupled nonlinear Schrödinger potential

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Abstract. We establish *a priori* conditions for a Gel'fand–Levitan (GL) integral using some results of the Fredholm theory. As consequence, we obtain a recovering formula for the potential of the coupled nonlinear Schrödinger equations. The remarkable fact is that the recovering formula is given in terms of the solutions of a classical GL-integral equation.

1. Introduction

We shall consider the integral equation system

$$f(x+p^{\pm}\zeta)+G(x,\zeta)\tau+\int_x^{\infty}G(x,\eta)f(\eta+p^{\pm}\zeta)\,\mathrm{d}\eta=0. \tag{1}$$

For a matrix-valued function defined on the real plane in the form

$$f=\begin{pmatrix} f_+ & 0_{p_+\times p_+} \\ 0_{p_-\times p_-} & f_- \end{pmatrix} \tag{2}$$

the value $f(x+p^{\pm}\zeta)$ is obtained by the formula

$$f(x+p^{\pm}\zeta)=\begin{pmatrix} f_+(x+p_-\zeta) & 0_{p_+\times p_+} \\ 0_{p_-\times p_-} & f_-(x+p_+\zeta) \end{pmatrix}.$$

Here, the constant $n\times n$-matrix τ $(n=p_-+p_+)$ is determined by the null and identity matrix blocks

$$\tau=\begin{pmatrix} 0_{p_+\times p_-} & \mathbb{I}_{p_+\times p_+} \\ \mathbb{I}_{p_-\times p_-} & 0_{p_-\times p_+} \end{pmatrix} \tag{3}$$

where $\mathbb{I}$ and 0 denote the identity and null matrices, respectively. The subscripts indicate their corresponding sizes. The solution to this integral system consists of an $n\times n$-matrix-valued function G defined on the real plane. This integral equation system is a generalization of the classical Gel'fand–Levitan (GL) one. For this reason, we refer to it as an integral system of GL type. They arise in the study of evolution equation systems by means of the scattering method.

The case $p_+ = 1$ is related to the evolution equation systems: the coupled nonlinear Schrödinger (NLS) equations ($n = 3$) and the vectorial nonlinear Schrödinger equation, both of which are used in the modelling of optical fibres. We restrict ourselves to this particular case. We encountered a solution given in terms of solutions of the classical GL-integral equation.

In order to simplify the discussion we shall assume that the matrix-valued functions $f_\pm$ in expression (2) have entries in the Schwartz class.

In the next section, we shall focus on *a priori* existence to show uniqueness of this kind of integral system. Using the Fredholm theory we shall show that the uniqueness of the solution is valid if and only if it exists. The essential step is to decouple the integral system into four integral subsystems. Writing the matrix solution in four blocks, we shall obtain four integral systems that only involve one and only one block of the solution. One of these independent subsystems will become the classical Gel'fand–Levitan integral equation.

With these results, we shall conclude that for these two-block Ablowitz–Kaup–Newell–Segur (AKNS) systems, skew-symmetric AKNS potentials may be recovered from the scattering matrix via a GL-integral equation system as in the 2×2 generic case. It is remarkable that this cannot be done for other AKNS systems.

2. Decoupling the GL-integral system

Write the matrix-valued unknown in four blocks

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A and D are $p_\pm \times p_\pm$ square matrices, respectively. Multiply the matrices G, τ, f and sum the results as indicated in the identity (1), then we have the four integral equation systems:

$$\begin{aligned}
&f_+(x + p\zeta) + B(x, \zeta) + \int_x^\infty A(x, z) f_+(z + p\zeta)\, \mathrm{d}z = 0 \\
&A(x, \zeta) + \int_x^\infty B(x, \eta) f_-(\eta + p^{-1}\zeta)\, \mathrm{d}\eta = 0 \\
&f_-(x + p^{-1}\zeta) + C(x, \zeta) + \int_x^\infty D(x, z) f_-(z + p^{-1}\zeta)\, \mathrm{d}z = 0 \\
&D(x, \zeta) + \int_x^\infty C(x, \eta) f_+(\eta + p\zeta)\, \mathrm{d}\eta = 0.
\end{aligned} \tag{4}$$

Note that each integral system involves at least two blocks. So our next goal is to obtain an integral system involving only one block. To do this, we solve the second and fourth equations for B and C, respectively. Then by substituting these results in the first and third equations, respectively, and by reversing the integration orders by Fubini's theorem, we obtain the integral equations which are integral systems of GL type for A and D:

$$\begin{aligned}
&A(x, \zeta) - \int_x^\infty f_+(x + p\eta) f_-(\eta + p^{-1}\zeta)\, \mathrm{d}\eta \\
&\qquad - \int_x^\infty A(x, z) \int_x^\infty f_+(x + p\eta) f_-(\eta + p^{-1}\zeta)\, \mathrm{d}\eta\, \mathrm{d}z = 0 \\
&D(x, \zeta) - \int_x^\infty f_-(x + p^{-1}\eta) f_+(\eta + p\zeta)\, \mathrm{d}\eta \\
&\qquad - \int_x^\infty D(\eta, z) \int_x^\infty f_-(x + p^{-1}\eta) f_+(\eta + p\zeta)\, \mathrm{d}z\, \mathrm{d}\eta = 0.
\end{aligned} \tag{5}$$

On the other hand, inverting the above procedure, i.e. solving for A and D and then replacing the results, we now obtain the Fredholm matrix equations

$$
\begin{aligned}
&f_+(x + p\zeta) + B(x, \zeta) - \int_x^\infty B(\eta, z) \int_x^\infty f_-(x + p^{-1}\eta) f_+(\eta + p\zeta)\, \mathrm{d}\eta\, \mathrm{d}z = 0 \\
&f_-(x + p^{-1}\zeta) + C(x, \zeta) - \int_x^\infty C(x, z) \int_x^\infty f_+(x + p\eta) f_-(\eta + p^{-1}\zeta)\, \mathrm{d}\eta\, \mathrm{d}z = 0.
\end{aligned} \tag{6}
$$

Since the entries of f are in the Schwartz class, the following condition is valid:

$$
\int_x^\infty (p_\pm u - x)|f_\pm(u)|^2\, \mathrm{d}u < \infty. \tag{7}
$$

Hence both integral operators

$$
F_x^\pm \mathcal{F}(s) = \int_x^\infty f_\pm(s + p^\pm z)\mathcal{F}(z)\, \mathrm{d}z
$$

are Hilbert Schmidt operators. Therefore, so are the compositions $F_x^+ F_x^-$ and $F_x^+ F_x^-$. Since Fredholm's alternative method is valid, the integral equation systems in (6) have solutions if and only if they are unique.

3. Main result

We now make use the fact that $p_+ = 1$. In this case, the first equation in (6) is the classical GL-integral equation for which the existence and uniqueness of the solution is well known. In fact, the solution is given in terms of infinite determinants. By the first identity in (4), we have the block B of G in terms of the A (the solution for the classical GL equation).

The results that have been obtained for the AKNS systems ensure the existence of the solution of the GL-integral system type given in (1) and consequently of those in (5) and (6). So we have uniqueness by the above discussion. On the other hand, if the matrix solution G is an *a priori* skew-symmetric condition

$$
C = -B^* \tag{8}
$$

then by the fourth relation in (4), D is given in terms of B. Consequently, G is given in terms of the solution of the GL classical integral equation.

By combining the fact that an AKNS potential induces a solution of the integral equation system of GL type (1), we have

Theorem. *Let q be a skew-symmetric two-block AKNS potential with entries in the Schwartz* *class, which induces the GL-integral equation system (1). Then*

$$q(x) = \left[J, \begin{pmatrix} 0 & -f_+\left(\frac{n}{n-1}x\right) \\ f_+^*\left(\frac{n}{n-1}x\right) & \int_x^\infty f_+^*\left(x + \frac{1}{n-1}\eta\right) f_+\left(\eta + \frac{1}{n-1}x\right) \mathrm{d}\eta \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} A(x,x) & -\int_x^\infty A(x,z) f_+\left(z + \frac{1}{n-1}\zeta\right) \mathrm{d}z \\ \int_x^\infty A^*(x,z) f_+^*\left(z + \frac{1}{n-1}\zeta\right) \mathrm{d}z & \int_x^\infty \int_x^\infty A^*(x,z) f_+^*\left(x + \frac{1}{n-1}\eta\right) \times f_+\left(\eta + \frac{1}{n-1}x\right) \mathrm{d}z\, \mathrm{d}\eta \end{pmatrix} \right] \tag{9}$$

where the brackets mean the commutator of matrices, $J = \mathrm{diag}(n-1, -1, \ldots, -1)$ *and* A *is the solution of the classical integral GL equation induced by the operator* $F_x^+ F_x^-$*:*

$$A(x, \zeta) = -\frac{D(x,\zeta)}{D(x)}$$

where

$$D(x) = \sum_{n=0}^{\infty} \int_x^\infty \ldots \int_x^\infty f_+ f_- \begin{pmatrix} t_1 & t_2 & \ldots & t_n \\ t_1 & t_2 & \ldots & t_n \end{pmatrix} \mathrm{d}t_1\, \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

$$D(x,\zeta) = \sum_{n=0}^{\infty} \int_x^\infty \ldots \int_x^\infty f_+ f_- \begin{pmatrix} x & t_1 & t_2 & \ldots & t_n \\ \zeta & t_1 & t_2 & \ldots & t_n \end{pmatrix} \mathrm{d}t_1\, \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

with

$$f_+ f_- \begin{pmatrix} s_1 & s_2 & \ldots & s_n \\ t_1 & t_2 & \ldots & t_n \end{pmatrix} = \det \left(\int_x^\infty f_+\left(s_j + \frac{1}{n-1}\eta\right) f_-(\eta + (n-1)t_k)\, \mathrm{d}\eta \right)_{jk}.$$

Remark 1. Here we have used the general recovering formula of the two-block AKNS potentials

$$q(x) = [J, G(x,x)].$$

In our case this formula justifies the *a priori* symmetry condition in (8) of the solution G that is essential to obtain the formula in (9).

Remark 2. Since q is off-J-diagonal, the expression in (8) can be substantially reduced as follows:

$$q(x) = \left[J, \begin{pmatrix} 0 & -f_+\left(\frac{n}{n-1}x\right) \\ f_+^*\left(\frac{n}{n-1}x\right) & 0_{n\times n} \end{pmatrix} + \int_x^\infty \begin{pmatrix} 0 & -A(x,z)f_+\left(z+\frac{1}{n-1}\zeta\right) \\ A^*(x,z)f_+^*\left(z+\frac{1}{n-1}\zeta\right) & 0_{n-1\times n-1} \end{pmatrix} \mathrm{d}z \right].$$

Remark 3. The recovering formula (8) depends not only upon $A(x,x)$ but also $A(x,\zeta)$. In contrast to the nonlinear Schrödinger equation the recovering formula only includes $A(x,x)$. In fact, due to the symmetry of the corresponding wavefunctions we have the nice formula in this case

$$q(x) = -2\frac{\mathrm{d}}{\mathrm{d}x}\arg\det(I+F_x^+F_x^-)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where we used the Dixon–Jost theorem

$$D(x,x) = \frac{\mathrm{d}}{\mathrm{d}x}D(x) = \frac{\mathrm{d}}{\mathrm{d}x}\det(I+F_x^+F_x^-).$$

The *a priori* symmetry condition (8) is not necessary because it is satisfied by the wavefunctions which is no longer true in the present case.

Remark 4. If

$$q = \begin{pmatrix} 0 & u & v \\ -\bar{u} & 0 & 0 \\ -\bar{v} & 0 & 0 \end{pmatrix}$$

then the flow induced is the coupled nonlinear Schrödinger equation:

$$\begin{aligned} u_t &= \mathrm{i}u_{xx} + 2\mathrm{i}u(|u|^2+|v|^2) \\ v_t &= \mathrm{i}v_{xx} + 2\mathrm{i}v(|u|^2+|v|^2). \end{aligned}$$

This type of equation has been studied by formulating a Riemann–Hilbert problem which is the natural formulation of the $\bar{\partial}$-method.

More generally, the potential

$$q = \begin{pmatrix} 0 & u_1 & \dots & u_{n-1} \\ -\bar{u}_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{u}_{n-1} & 0 & \dots & 0 \end{pmatrix}$$

produces the vectorial nonlinear Schrödinger equation

$$\boldsymbol{u}_t = \mathrm{i}\boldsymbol{u}_{xx} + (n-1)\mathrm{i}|\boldsymbol{u}|^2\boldsymbol{u}$$

where $\boldsymbol{u} = (u_1, \dots, u_{n-1})$.

4. Conclusions

The coupled nonlinear Schrödinger equation has been investigated via a Riemann–Hilbert formulation. The number of investigations using the GL formulation for such an evolution system has decreased over the last few years. However, the GL formulation is still of some interest due to its relatively complicated structure.

Using a GL formulation, similar results has been obtained for highly nonlinear Schrödinger equations without including the coupled nonlinear Schrödinger equations, but the remarkable thing here is that the recovering solution is expressed in terms of a solution of the GL classical integral equation. This is in contrast to the Riemann–Hilbert formulation for which the solution has a complicated structure, even for symmetric potentials.

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